NPS55-81-023PR

NAVAL POSTGRADUATE SCHOOL

Monterey, California

SUMMARY REPORT FOR 1981

by

Alan R. Washburn

and

Bruno O. Shubert

September 1981

Approved for public release; distribution unlimited

Prepared for:
Naval Underwater Systems Center
Newport, RI 02840

NAVAL POSTGRADUATE SCHOOL
MONTEREY, CALIFORNIA

Rear Admiral J. J. Ekelund
Superintendent

D. A. Schrady
Acting Provost

This report was prepared by:

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: NPS55-81-023PR	2. GOVT ACCESSION NO.
3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle): SUMMARY REPORT FOR 1981	5. TYPE OF REPORT & PERIOD COVERED: Project Report
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): Alan R. Washburn Bruno O. Shubert	8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Naval Postgraduate School Monterey, CA 93940	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 62711N ; SF11125491 N6660481WR10106
11. CONTROLLING OFFICE NAME AND ADDRESS: Naval Underwater Systems Center Newport, RI 02840	12. REPORT DATE: September 1981
	13. NUMBER OF PAGES: 53
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	15. SECURITY CLASS. (of this report): UNCLASSIFIED
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Least Squares
ASW
Detection

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This report summarizes work done for NUSC in 1981 on the topic of statistical estimation problems in ASW.

Summary Report for 1981

A. R. Washburn

B. O. Shubert

Abstract: This report summarizes work done for NUSC in 1981 on the topic of statistical estimation problems in ASW.

Keywords: Least Squares, ASW, Detection

Introduction:

Most of our work in 1981 depends on the observation that, if processing time is not important, better statistical estimates can be made by minimizing a least squares type objective function by trying all possibilities, rather than by employing non-linear transformations to permit the minimization to be done analytically. The first two sections document the extent of the improvement in various circumstances. The improvement is equivalent to several decibels gain in signal-to-noise ratio.

Section three documents an idea for determining direction from three measurements. The procedure does not require initial estimates of speed, range, or signal strength.

Section four is unrelated to the first three sections. It uses an equivalent sweep width idea to emphasize the difference between area search and barrier patrol.

1. Performance of various least squares objectives

1.1 Background

Consider a situation where a sequence of measurements C_1, C_2,... of some phenomenon are made, each of which depends in theory on a set of parameters $\underline{p}$ as well as some known independent variables. If the parameters $\underline{p}$ are unknown, it is natural to estimate them by adopting the ones that make theoretical and actual measurements agree as closely as possible. The resulting estimates, as well as the difficulty of obtaining them, depend on what is meant by "closely". The purpose of this section is to explore the tradeoff between accuracy and difficulty that results from different definitions of the word in a particular estimation problem. Specifically, we assume that the theoretical ith measurement is

$$(1) \qquad \bar{C}_i = K|x + vt_i - x_i|^{-\alpha} \text{, where}$$

v and x are the speed and initial distance to a target, x_i and t_i are the position and time at which the sensor obtains the ith measurement, and where K and α are two positive parameters that will be assumed known initially. So $\underline{p} = (x,v)$ and x_i and t_i are the independent variables.

1.2 Direct Search Method

A natural way to estimate $\underline{p}$ would be to minimize the Euclidean distance between $(C_1,\ldots,C_n)$ and $(\bar{C}_1,\ldots,\bar{C}_n)$, where

n is the number of measurements. So we define

$$(2) \qquad E_n(x,v) = \sum_{i=1}^{n} (C_i - \bar{C}_i)^2 \text{ , and let}$$

(x_{DIR}, v_{DIR}) be the point that minimizes $E_n(x,v)$. If the only difference between C_i and $\bar{C}_i$ is one of additive white noise, then x_{DIR} and v_{DIR} have the virtues of being maximum likelihood estimates of x and v. Since the function $E_n(x,v)$ has no easily exploitable analytic properties other than being a sum, x_{DIR} and v_{DIR} are most easily obtained by direct search of some set S (typically a grid) of values for (x,v). With each measurement, the stored values $E_n(x,v)$ are updated by employing the equation

$$(3) \qquad E_{n+1}(x,v) = E_n(x,v) + (C_{n+1} - \bar{C}_{n+1})^2 \text{ ,}$$

and then x_{DIR} and v_{DIR} are selected by finding the smallest stored value. The disadvantage of this method is evidently one of time, particularly if S is a large set (fine grid). The advantages are that $E_n(x,v)$ is essentially the right thing to minimize when additive noise is the dominant difference between C_i and $\bar{C}_i$, and that the amount of time required per estimate is independent of the number of measurements that have been made.

1.3 Least Squares Estimates

Let $y_i = C_i^{-1/\alpha}$, $\bar{y}_i = \bar{C}_i^{-1/\alpha} = K^{-1/\alpha}|x + vt_i - x_i|$, and

$$F_n(x,v) = \sum_{i=1}^{n} (y_i - \bar{y}_i)^2 . \tag{4}$$

In circumstances where the sign of $x + vt_i - x_i$ is known and constant for all observations, $\bar{y}_i$ is a linear function of x and v, and $F_n(x,v)$ can be minimized analytically. In fact, the minimizing point (x_{LSQ}, v_{LSQ}) can be obtained using standard linear regression formulas. We will refer to x_{LSQ} and v_{LSQ} as "least squares estimates", even though x_{DIR} and v_{DIR} are equally deserving of the name. The advantage of the least squares estimates is the simplicity of obtaining them. The disadvantages are that they are not maximum likelihood if it is the quantity C_i (rather than y_i) that suffers from additive noise, and that the target's direction of travel must be known.

1.4 A Comparison of Least Squares and Direct Search Estimates

A simulation was performed to compare (x_{LSQ}, v_{LSQ}) with (x_{DIR}, v_{DIR}) . The true (x,v) was (30NM, 5KT) in all cases, with $K = \alpha = 1$. The independent variables were $t_i = i/6$ and $(x_1, x_2, \ldots) = (2,2,4,4,6,\ldots)$, which corresponds to jerky sensor motion at an average speed of 6KT. The measured C_i were obtained from $\bar{C}_i$ by

(5) $$C_i = |\bar{C}_i \exp(A_1W_1) + A_2W_2| ,$$

where W_1 and W_2 are unit normal samples and A_1 and A_2 are parameters. The additive noise is governed by A_2, and the multiplicative noise by A_1. The reason for introducing multiplicative noise was to test robustness of the two estimators to the form of the noise.

The DIR procedure was programmed to examine only the possibilities 1,2,..., 50 for x and .5,1.0,...,15 for v - a total of 1500 possibilities - from which the best was selected. There is no reason in principle why the DIR procedure should not also search negative values and thereby determine the sign as well as the magnitude of x and v. In fact, the original version of the procedure did exactly that. Initial runs with this version revealed that it sometimes got the sign wrong, particularly with only 2 or 3 measurements. While this was not surprising, it was surprising (initially) that X_{LSQ} did not suffer similarly. But the LSQ procedure does not mistake the sign because an assumption about sign is intrinsic to the procedure, and the assumption happens to be correct in the simulation; the LSQ procedure showed no capability for estimating (x,v) when that assumption was wrong. The "fair" thing to do seemed to be to give the correct direction to both procedures, which is why DIR searched only positive values for x and v even though this would not be true in an operational version.

It was also found that the LSQ procedure occasionally produced wild estimates that would simply be dismissed in practice, but which significantly affected the summary statistics. The remedy for this was to make 0 estimates if X_{LSQ} or V_{LSQ} exceeded 100 or 50 in absolute value.

Figures 1-16 show the results of comparing the LSQ and DIR procedures. Each figure displays the experimental mean and standard deviation (in the +/- column) of the four estimates X_{LSQ}, V_{LSQ}, X_{DIR}, and V_{DIR}, on each of 20 successive measurements. All statistics are based on 50 independent replications, so a 68% confidence interval would have a width of whatever is in the +/- column divided by $\sqrt{50} \approx 7$.

Figures 1-8 show performance with no multiplicative error and increasing amounts of additive error. The scale of A2 was chosen to cover the range from good performance to disaster; with different values for v and x, a different scale would be needed. When A2 = .002, both procedures obtain good estimates on the average. Note, however, that the LSQ procedure has a higher standard deviation. On the second measurement, for example, X_{LSQ} has a standard deviation of 3.98, which considerably exceeds 1.88 for X_{DIR}. Neither procedure is of much use for estimating velocity after only two measurements. As A2 is increased, the fact that LSQ tends to have larger standard deviations than DIR is accentuated. By the time A2 = .02, LSQ is probably useless, whereas DIR might still have some value for estimating x or even v with enough observations.

Figures 9-12 show results with A2 = 0 and increasing values of A1. Again, LSQ seems to have greater standard deviations than DIR.

Figures 13 and 14 show results with mixed error types.

Figures 15 and 16 show results where the true X is 30.5, which doesn't happen to coincide with any point that DIR searches. In figure 15, X_{DIR} has a larger standard deviation that X_{LSQ}, presumably because DIR is always at least .5 away from the truth. Figure 16 shows that this effect is negligible when measurement errors are larger.

TRUE X= 30.0000 V= 5.0000 A1= 0.0 A2= 0.0020

XLSQ	+/-	VLSQ	+/-	XDIR	+/-	VDIR	+/-
0.0	0.0	0.0	0.0	30.26	1.69	3.45	1.55
29.96	3.95	4.27	14.35	29.68	1.88	6.18	6.03
30.04	2.33	4.39	6.04	29.76	1.82	5.49	4.58
30.11	2.02	4.57	3.95	29.92	1.90	5.01	3.63
30.07	1.81	4.77	2.32	30.00	1.82	4.81	2.75
29.95	1.53	5.08	2.16	29.98	1.63	5.13	2.26
30.00	1.37	4.78	1.55	29.88	1.34	5.07	1.54
29.96	1.25	5.05	1.34	29.80	1.18	5.19	1.29
29.98	1.17	5.02	1.13	29.92	1.15	5.06	1.07
29.99	1.05	5.00	0.86	29.88	0.97	5.05	0.77
29.98	1.00	5.02	0.73	29.82	0.89	5.11	0.59
29.99	0.96	5.00	0.63	29.80	0.94	5.12	0.56
29.98	0.90	5.02	0.53	29.86	0.96	5.07	0.55
29.97	0.86	5.02	0.47	29.90	0.85	5.08	0.44
29.99	0.84	5.00	0.42	30.00	0.75	5.00	0.37
29.99	0.81	5.00	0.39	29.98	0.65	5.01	0.32
29.99	0.78	5.00	0.35	30.04	0.40	4.98	0.20
29.99	0.75	5.00	0.32	30.00	0.20	5.00	0.10
29.99	0.73	5.00	0.29	30.00	0.03	5.00	0.0
29.99	0.70	5.00	0.27	30.00	0.03	5.00	0.0

Figure 1

TRUE X= 30.0000 V= 5.0000 A1= 0.0 A2= 0.0050

XLSQ	+/-	VLSQ	+/-	XDIR	+/-	VDIR	+/-
0.0	0.0	0.0	0.0	30.74	4.55	2.99	1.80
30.17	10.93	0.87	21.25	29.63	3.46	6.49	6.74
30.62	6.20	4.41	16.05	29.50	3.22	6.01	6.00
30.78	5.29	3.82	10.27	29.56	3.34	5.53	5.63
30.73	4.75	3.08	7.23	29.38	3.27	5.69	4.79
30.39	4.20	4.35	5.68	29.44	3.27	5.31	4.44
30.48	3.69	4.55	4.19	29.84	2.89	5.04	3.12
30.36	3.37	4.39	3.59	29.60	2.71	5.41	2.95
30.39	3.13	4.83	3.02	29.74	2.54	5.21	2.47
30.41	2.80	4.80	2.29	29.68	2.33	5.19	1.85
30.39	2.66	4.34	1.94	29.62	2.24	5.25	1.45
30.41	2.53	4.79	1.67	29.82	2.31	5.05	1.43
30.37	2.37	4.35	1.40	29.68	1.93	5.21	1.03
30.34	2.29	4.88	1.26	29.75	1.85	5.17	0.90
30.38	2.21	4.35	1.12	29.96	2.00	5.04	0.90
30.37	2.14	4.36	1.02	29.96	1.92	5.02	0.85
30.37	2.07	4.35	0.93	29.94	1.95	4.99	0.75
30.36	1.99	4.86	0.84	29.80	1.66	5.06	0.61
30.35	1.92	4.37	0.77	29.34	1.45	5.04	0.48
30.35	1.85	4.37	0.71	29.70	1.32	5.09	0.41

Figure 2

TRUE X= 30.0000 V= 5.0000 A1= 0.0 A2= 0.0100

XLSQ	+/-	VLSQ	+/-	XDIR	+/-	VDIR	+/-
0.0	0.0	0.0	0.0	31.60	8.40	4.48	3.17
27.35	21.16	-0.34	22.23	30.34	6.48	5.52	6.92
31.71	13.94	3.26	22.45	29.36	5.35	5.57	6.55
32.74	13.69	1.18	20.83	29.52	5.50	5.78	6.17
34.79	14.76	2.48	15.35	28.62	4.95	6.56	5.90
33.82	13.38	3.28	14.70	28.34	4.38	6.75	5.81
33.81	12.05	2.13	13.23	29.20	4.45	5.79	4.55
33.33	10.90	3.15	11.14	29.00	4.39	6.11	4.46
33.32	10.04	3.16	9.31	29.14	4.19	5.34	4.01
33.27	8.95	3.24	7.15	29.34	4.34	5.57	3.49
33.18	8.39	3.38	6.07	29.48	4.22	5.43	2.82
33.16	7.83	3.40	5.16	29.72	4.29	5.16	2.72
32.99	7.35	3.52	4.41	29.50	3.34	5.38	2.12
32.86	7.03	3.78	3.96	29.54	3.75	5.32	1.89
32.88	6.73	3.76	3.53	29.98	4.01	4.93	1.76
32.80	6.46	3.35	3.18	29.94	3.96	5.00	1.69
32.76	6.20	3.38	2.83	29.96	3.42	4.95	1.35
32.68	5.94	3.96	2.61	29.82	2.88	5.03	1.06
32.61	5.68	4.02	2.36	29.74	2.49	5.08	0.84
32.55	5.47	4.07	2.16	29.84	2.44	5.05	0.80

Figure 3

TRUE X= 30.0000 V= 5.0000 A1= 0.0 A2= 0.0150

XLSQ	+/-	VLSQ	+/-	XDIR	+/-	VDIR	+/-
0.0	0.0	0.0	0.0	32.34	10.79	4.94	4.37
25.52	25.58	-1.66	16.77	31.10	9.66	6.70	7.03
31.50	21.89	1.91	18.33	30.16	8.25	6.95	6.72
31.27	20.24	-0.41	20.18	29.40	7.26	6.36	6.64
31.73	18.72	-0.42	22.27	28.56	6.79	7.03	6.21
30.45	17.58	4.39	20.44	28.54	6.39	7.30	6.16
30.75	16.76	2.75	15.26	28.50	5.98	6.77	5.62
30.58	15.72	4.09	13.70	28.50	5.87	6.86	5.50
29.36	22.03	3.13	10.71	28.50	5.40	6.52	4.85
30.75	16.19	1.91	10.79	28.86	5.73	6.06	4.42
31.52	13.86	1.92	9.25	29.02	5.78	5.85	3.85
34.13	15.25	1.30	7.85	29.46	5.84	5.37	3.60
34.26	14.85	3.00	8.80	29.22	5.36	5.63	3.10
34.30	14.99	2.00	9.88	29.24	5.31	5.62	2.90
34.56	15.24	1.33	8.39	29.96	5.13	4.97	2.29
34.63	15.52	1.36	7.46	29.82	5.20	5.04	2.27
34.69	15.66	1.88	6.76	30.06	4.97	4.90	1.94
34.66	15.70	1.98	6.23	29.74	4.35	5.04	1.61
34.55	15.46	2.09	5.79	29.46	3.93	5.18	1.34
34.39	14.98	2.20	5.41	29.60	3.53	5.11	1.15

Figure 4

TRUE X= 30.0000 V= 5.0000 A1= 0.0 A2= 0.0200

XLSQ	+/-	VLSQ	+/-	XDIR	+/-	VDIR	+/-
0.0	0.0	0.0	0.0	32.54	12.23	5.66	4.97
21.67	34.31	-2.03	15.23	30.82	10.94	7.27	7.10
28.34	23.85	0.19	19.56	30.04	9.85	7.41	6.79
29.05	33.32	-1.31	18.03	29.34	8.87	6.41	6.78
31.05	25.70	0.20	19.08	28.14	7.98	7.45	6.46
24.97	28.89	3.59	20.95	28.18	7.40	7.70	6.26
23.90	27.40	2.46	14.23	28.08	6.96	7.13	6.07
26.42	22.84	4.32	15.56	27.72	6.74	7.73	5.98
27.08	19.51	5.31	12.17	27.33	6.52	7.16	5.37
33.95	23.44	4.56	17.51	28.29	6.94	6.53	5.00
36.48	28.65	0.54	15.97	28.24	6.95	6.50	4.68
36.12	25.35	-0.27	14.04	28.88	7.12	5.83	4.44
36.67	23.35	-1.66	14.16	28.84	6.64	5.92	3.69
37.13	22.85	-2.15	13.64	28.80	6.37	5.36	3.42
37.79	22.92	-1.88	14.33	29.64	6.14	5.09	2.72
38.04	23.27	-2.23	13.10	29.44	6.13	5.23	2.67
36.19	22.24	-2.46	12.31	29.92	5.97	4.96	2.36
36.07	21.84	-3.43	13.41	29.86	5.28	5.01	1.96
35.91	21.47	-3.22	12.71	29.44	4.77	5.20	1.65
35.76	21.09	-2.99	12.07	29.56	4.30	5.12	1.40

Figure 5

TRUE X= 30.0000 V= 5.0000 A1= 0.0 A2= 0.0300

XLSQ	+/-	VLSQ	+/-	XDIR	+/-	VDIR	+/-
0.0	0.0	0.0	0.0	32.06	14.15	6.54	5.22
16.70	36.38	-0.03	11.61	29.34	12.52	7.29	6.97
21.47	30.73	-0.32	17.76	28.80	11.01	7.58	6.81
22.17	30.82	-3.73	17.90	27.84	10.24	6.73	6.79
23.32	31.29	-0.31	21.29	26.68	9.44	7.92	6.55
23.32	31.40	2.17	19.00	27.02	9.12	8.26	6.56
28.93	25.52	4.98	16.79	26.46	8.47	7.94	6.32
28.80	24.61	3.36	17.87	26.54	8.32	8.11	6.30
30.44	24.39	3.08	18.19	26.34	7.96	8.11	6.03
30.77	27.42	3.72	17.05	26.60	8.16	7.42	5.39
30.70	28.63	4.57	17.32	26.28	8.44	7.60	5.46
33.51	26.56	1.51	14.05	26.88	8.55	6.81	5.03
34.84	25.39	3.10	15.08	26.76	8.17	7.04	4.40
33.43	30.71	2.51	17.01	26.92	7.85	6.37	4.18
33.49	27.38	1.05	13.95	28.04	7.27	5.31	3.36
34.57	26.03	-0.44	13.68	27.76	7.28	5.39	3.28
35.66	25.63	-0.93	12.23	28.40	7.55	5.51	2.98
34.29	23.94	-0.11	13.16	28.62	6.60	5.45	2.49
34.72	23.55	-1.31	13.19	28.04	5.91	5.68	2.07
35.13	23.23	-1.45	11.74	28.58	5.52	5.44	1.83

Figure 6

TRUE X= 30.0000 V= 5.0000 A1= 0.0 A2= 0.0500

XLSQ	+/-	VLSQ	+/-	XDIR	+/-	VDIR	+/-
0.0	0.0	0.0	0.0	29.58	15.80	6.24	5.44
12.08	30.97	-0.45	13.08	26.14	14.26	7.98	6.99
16.08	32.97	-2.07	11.60	25.12	12.12	7.78	6.85
22.91	26.77	1.06	14.51	23.32	10.98	7.10	6.68
23.24	28.86	3.00	15.95	22.12	8.94	8.58	6.60
24.54	31.65	0.95	19.16	22.42	8.72	8.73	6.54
25.07	25.43	2.01	20.82	22.06	8.06	8.46	6.38
24.03	34.02	2.31	20.58	22.26	8.19	8.44	6.49
24.59	31.11	3.20	17.55	22.12	7.29	8.61	5.80
26.14	25.66	1.88	16.09	22.04	7.34	8.44	5.39
29.64	25.46	2.31	19.43	21.84	8.25	8.36	5.23
31.16	25.74	2.67	16.22	22.06	8.32	8.37	5.28
29.17	29.30	5.91	16.85	21.68	8.03	8.91	4.81
32.44	29.76	4.19	12.67	21.78	8.04	8.70	4.32
30.33	27.22	3.88	12.80	22.98	8.20	7.64	4.16
27.18	33.48	2.19	12.89	23.26	8.60	7.53	4.18
29.44	29.60	0.36	13.13	23.66	8.14	7.18	3.46
23.66	32.49	0.58	11.50	23.30	6.84	7.31	2.81
26.66	27.71	0.22	14.12	23.30	6.31	7.28	2.57
28.97	25.27	-0.07	12.55	23.88	6.49	6.98	2.26

Figure 7

TRUE X= 30.0000 V= 5.0000 A1= 0.0 A2= 0.1000

XLSQ	+/-	VLSQ	+/-	XDIR	+/-	VDIR	+/-
0.0	0.0	0.0	0.0	22.52	16.09	5.66	4.81
5.40	24.71	-2.52	11.55	16.50	10.96	6.94	6.96
11.51	25.00	0.09	14.30	16.18	10.04	8.47	6.47
23.48	27.87	1.47	18.97	14.68	7.55	6.97	6.31
19.96	29.41	5.64	18.46	13.64	5.66	9.29	6.24
19.23	31.62	3.32	17.27	13.74	5.90	9.23	6.17
22.90	29.94	4.83	16.23	13.52	5.28	9.96	5.58
22.28	30.91	5.63	17.93	13.42	5.12	10.01	5.22
22.46	31.07	7.44	20.74	13.68	5.41	10.07	4.87
16.46	28.95	4.42	18.54	13.38	5.43	10.10	4.84
15.56	29.20	7.81	19.83	12.62	4.67	11.12	3.32
18.25	29.00	4.04	20.22	13.28	6.00	10.47	3.98
24.71	26.09	3.44	17.76	13.16	5.96	10.60	3.75
25.50	28.45	3.70	16.63	13.22	5.68	10.51	3.35
26.43	27.32	7.60	16.80	13.68	6.39	10.17	3.45
28.49	33.85	7.31	14.10	13.42	6.29	10.32	3.30
31.51	26.70	7.36	16.26	13.26	5.21	10.32	2.52
30.66	25.53	6.42	13.66	13.10	4.77	10.39	2.29
32.38	24.88	6.99	13.32	13.20	4.35	10.29	1.93
31.30	26.59	6.24	11.83	13.70	4.47	10.02	1.85

Figure 8

TRUE X= 30.0000 V= 5.0000 A1= 0.1000 A2= 0.0

XLSQ	+/-	VLSQ	+/-	XDIR	+/-	VDIR	+/-
0.0	0.0	0.0	0.0	30.26	2.09	3.45	1.57
29.49	5.17	5.56	19.69	29.53	2.04	7.15	6.58
29.81	3.34	6.05	10.11	29.40	2.09	7.07	5.95
30.57	2.68	5.90	5.46	29.86	2.09	5.15	4.73
30.37	2.55	4.53	4.25	29.94	2.16	5.01	3.46
30.38	2.50	4.49	3.49	30.04	2.42	4.72	3.43
30.21	2.29	4.37	2.81	29.83	2.09	5.12	2.68
30.16	2.02	4.99	2.18	29.73	1.80	5.24	1.99
30.18	1.95	4.95	1.86	29.90	2.04	5.06	1.91
30.29	1.77	4.75	1.50	30.10	1.64	4.30	1.36
30.14	1.66	4.99	1.27	29.90	1.72	5.11	1.44
30.09	1.55	5.05	1.03	29.82	1.51	5.17	1.07
30.14	1.45	4.78	0.89	30.02	1.36	4.99	0.86
30.18	1.40	4.94	0.80	30.12	1.44	4.91	0.89
30.16	1.35	4.96	0.71	29.96	1.56	5.02	0.81
30.14	1.29	4.98	0.63	30.02	1.32	5.04	0.62
30.17	1.26	4.96	0.59	30.13	1.76	4.90	0.80
30.17	1.25	4.95	0.56	30.26	1.59	4.87	0.68
30.18	1.22	4.95	0.52	30.24	1.48	4.90	0.58
30.16	1.16	4.96	0.46	29.83	1.52	5.03	0.55

Figure 9

TRUE X= 30.0000 V= 5.0000 A1= 0.2000 A2= 0.0

XLSQ	+/-	VLSQ	+/-	XDIR	+/-	VDIR	+/-
0.0	0.0	0.0	0.0	30.46	3.98	3.19	1.77
28.92	10.49	-1.09	23.94	29.50	3.52	7.55	6.91
29.71	6.72	7.32	19.84	29.38	3.13	7.74	6.69
31.30	5.35	5.11	12.95	29.46	3.23	5.64	5.84
30.93	5.08	4.21	3.62	29.16	3.41	5.34	5.12
31.02	5.08	4.60	7.11	29.42	3.67	5.17	4.85
30.71	4.67	4.69	5.76	29.08	3.43	5.37	4.31
30.61	4.09	4.90	4.44	29.30	3.44	5.65	3.72
30.65	3.97	4.31	3.77	29.36	3.45	5.42	3.29
30.89	3.62	4.42	3.06	29.96	3.09	4.69	2.65
30.56	3.40	4.92	2.61	29.30	3.18	5.46	2.53
30.47	3.15	5.05	2.12	29.13	2.70	5.58	1.89
30.58	2.93	4.90	1.84	29.74	2.72	5.07	1.77
30.65	2.86	4.32	1.65	29.98	2.87	4.38	1.69
30.62	2.75	4.35	1.47	29.74	3.05	5.08	1.57
30.58	2.64	4.90	1.30	29.54	2.57	5.15	1.22
30.62	2.58	4.35	1.20	30.22	3.57	4.30	1.69
30.64	2.55	4.34	1.15	29.88	2.73	4.95	1.21
30.65	2.48	4.33	1.06	29.94	2.70	4.94	1.06
30.62	2.37	4.36	0.94	29.60	2.62	5.10	0.95

Figure 10

TRUE X= 30.0000 V= 5.0000 A1= 0.5000 A2= 0.0

XLSQ	+/-	VLSQ	+/-	XDIR	+/-	VDIR	+/-
0.0	0.0	0.0	0.0	31.44	9.73	4.03	3.23
26.55	28.72	-3.52	19.11	29.64	8.64	7.94	7.05
29.83	13.2[illegible]	2.04	22.91	29.08	7.83	8.10	7.07
34.59	14.34	0.45	22.54	27.73	7.13	6.39	6.53
33.94	13.71	2.98	19.67	26.90	6.59	6.56	5.97
34.65	13.99	2.56	19.79	27.08	6.42	5.53	5.98
34.05	13.06	3.92	16.20	26.24	5.70	7.25	5.87
33.91	11.47	4.21	12.37	26.40	5.97	7.21	5.43
34.09	11.22	3.88	10.52	26.12	5.01	7.13	5.23
34.72	10.41	2.85	8.64	27.16	5.58	5.77	4.70
33.66	9.97	4.44	7.63	26.42	6.01	6.31	4.53
33.48	9.19	4.59	6.19	26.38	5.76	6.34	3.86
33.87	8.73	4.19	5.43	27.10	5.97	6.10	3.75
33.96	8.41	4.08	4.97	27.94	5.99	5.30	3.43
33.95	8.05	4.10	4.37	27.08	5.75	5.91	2.97
33.80	7.75	4.25	3.89	27.60	5.95	5.63	2.82
33.91	7.49	4.14	3.54	28.00	6.59	5.35	3.07
33.97	7.35	4.08	3.33	28.12	6.25	5.28	2.78
34.04	7.16	4.02	3.09	27.52	5.93	5.33	2.27
33.94	6.82	4.11	2.71	26.94	5.92	5.75	2.12

Figure 11

TRUE X= 30.0000 V= 5.0000 A1= 1.0000 A2= 0.0

XLSQ	+/-	VLSQ	+/-	XDIR	+/-	VDIR	+/-
0.0	0.0	0.0	0.0	31.80	14.43	5.33	5.06
13.06	38.36	1.15	12.52	28.24	13.60	8.46	7.14
19.54	41.69	-0.37	15.49	26.86	12.83	8.29	6.96
30.62	24.54	-2.41	22.12	23.94	11.78	6.77	6.82
32.29	31.63	1.01	18.78	21.86	10.41	7.17	6.30
36.68	33.59	2.08	20.91	21.80	9.98	6.08	6.23
33.55	28.53	3.35	22.48	20.80	8.75	3.44	6.29
36.54	27.52	0.75	23.20	20.66	8.64	8.68	5.71
41.33	29.36	-1.50	21.93	19.78	8.08	8.85	5.85
41.05	27.35	-2.78	23.10	20.78	7.91	7.56	5.66
37.35	26.48	3.13	19.53	20.14	8.40	8.60	5.26
37.35	23.08	2.47	16.62	20.12	8.38	8.71	4.85
38.91	22.29	-0.10	16.29	20.56	8.70	8.18	4.60
38.90	21.97	1.22	17.07	20.98	8.70	7.36	4.73
39.50	21.24	0.83	14.74	20.32	8.79	7.80	4.31
41.24	21.54	1.31	13.22	20.74	9.05	7.42	3.93
41.57	20.62	1.07	11.78	21.23	9.12	7.16	4.12
42.01	20.37	0.77	10.79	21.14	8.99	7.02	3.98
42.37	20.06	0.53	9.99	21.46	8.70	6.88	3.56
42.25	19.30	0.78	8.83	21.30	8.96	6.92	3.60

Figure 12

TRUE X= 30.0000 V= 5.0000 A1= 0.2000 A2= 0.0100

XLSQ	+/-	VLSQ	+/-	XDIR	+/-	VDIR	+/-
0.0	0.0	0.0	0.0	31.94	9.25	3.27	2.84
28.92	22.07	-0.48	17.00	30.40	7.62	7.03	6.80
27.13	17.54	-1.48	19.60	30.46	7.47	6.22	6.64
31.10	15.07	2.01	20.53	29.22	6.23	5.90	6.46
31.71	15.42	2.47	18.46	28.66	5.97	6.22	6.09
33.09	16.70	3.46	15.58	28.42	5.39	6.38	5.97
33.09	15.44	2.04	15.13	28.34	4.97	6.57	5.00
32.58	14.43	3.06	13.07	28.22	4.99	6.79	5.25
32.64	13.79	2.95	11.36	28.26	4.57	6.47	4.37
32.93	12.72	2.47	8.75	29.04	4.73	5.44	3.73
32.50	11.93	3.12	7.18	28.70	5.01	5.93	3.33
32.42	11.54	3.23	6.36	28.92	5.06	5.66	3.05
32.36	11.04	3.30	5.51	29.03	4.48	5.55	2.56
32.25	10.77	3.44	5.01	29.50	4.52	5.14	2.34
32.24	10.33	2.55	3.05	29.64	4.43	5.06	2.08
32.11	10.12	2.76	7.30	29.56	4.54	5.12	2.04
32.13	9.90	2.35	6.57	30.02	4.44	4.36	1.94
32.07	9.64	2.98	5.96	29.64	3.99	5.00	1.59
32.02	9.43	3.09	5.41	29.44	3.93	5.11	1.40
31.92	9.13	3.24	4.90	29.22	3.46	5.20	1.20

Figure 13

TRUE X= 30.0000 V= 5.0000 A1= 0.2000 A2= 0.0200

XLSQ	+/-	VLSQ	+/-	XDIR	+/-	VDIR	+/-
0.0	0.0	0.0	0.0	32.74	12.91	5.80	5.35
21.41	34.69	0.34	14.77	30.34	11.62	7.78	7.09
20.08	32.63	1.07	16.25	30.16	10.34	6.77	6.95
24.58	23.33	-0.55	17.49	28.88	9.27	6.49	6.94
27.93	22.51	-0.15	17.69	28.20	8.73	6.75	6.53
23.83	29.30	5.28	13.20	27.94	8.40	7.37	6.33
25.71	27.92	1.99	15.36	27.68	7.70	7.20	5.82
27.89	23.34	2.16	16.81	27.42	7.24	7.77	6.18
31.26	24.97	3.33	18.91	27.00	6.84	7.77	5.35
35.43	24.90	2.29	16.81	27.66	7.22	6.72	5.27
35.26	27.34	3.17	14.97	27.26	7.26	7.15	4.84
37.13	24.17	2.18	12.03	27.82	7.25	6.44	4.45
37.69	23.88	1.46	10.26	28.12	6.88	6.25	3.88
34.43	30.71	0.36	11.04	28.50	6.82	5.88	3.58
36.85	25.85	-0.10	9.86	29.26	6.44	5.22	2.94
36.36	23.32	-0.06	9.05	29.16	6.69	5.27	2.92
37.71	23.91	-0.21	8.46	29.90	6.03	4.88	2.71
36.04	22.23	0.58	13.73	29.40	5.95	5.06	2.25
36.23	21.30	0.23	11.38	29.18	5.48	5.18	1.94
36.32	21.45	-0.02	9.88	28.96	5.05	5.28	1.69

Figure 14

TRUE X= 30.5000 V= 5.0000 A1= 0.0 A2= 0.0010

XLSQ	+/-	VLSQ	+/-	XDIR	+/-	VDIR	+
0.0	0.0	0.0	0.0	30.78	0.94	3.19	1.
30.46	2.03	5.11	5.01	30.23	1.36	5.78	5.
30.49	1.20	4.96	3.11	30.40	1.17	5.22	3.
30.52	1.03	4.35	2.04	30.52	1.14	4.33	2.
30.50	0.94	4.91	1.47	30.42	0.93	5.05	1.
30.44	0.81	5.06	1.11	30.46	0.88	5.04	1.
30.47	0.71	5.01	0.79	30.40	0.69	5.04	0.
30.45	0.65	5.04	0.69	30.40	0.57	5.07	0.
30.46	0.60	5.02	0.58	30.36	0.63	5.06	0.
30.47	0.54	5.02	0.45	30.36	0.59	5.09	0.
30.46	0.52	5.02	0.37	30.42	0.72	5.04	0.
30.47	0.50	5.01	0.32	30.54	0.64	4.92	0.
30.46	0.46	5.02	0.27	30.42	0.53	5.07	0.
30.46	0.45	5.02	0.24	30.40	0.53	5.10	0.
30.47	0.44	5.01	0.22	30.38	0.69	5.04	0.
30.47	0.42	5.01	0.20	30.44	1.02	5.04	0.
30.47	0.41	5.01	0.18	30.50	1.47	5.00	0.
30.47	0.39	5.01	0.16	30.50	1.50	5.00	0.
30.47	0.38	5.01	0.15	30.50	1.50	5.00	0.
30.47	0.37	5.01	0.14	30.38	1.50	5.04	0.

Figure 15

TRUE X= 30.5000 V= 5.0000 A1= 0.0 A2= 0.0100

XLSQ	+/-	VLSQ	+/-	XDIR	+/-	VDIR	+
0.0	0.0	0.0	0.0	32.14	8.40	4.28	3.
27.78	22.09	-3.57	20.09	30.80	6.74	6.59	6.
32.39	14.68	2.19	22.28	30.30	6.08	6.62	6.
33.51	14.56	1.04	19.19	30.02	5.70	5.30	6.
35.69	15.83	2.35	16.06	29.26	5.07	6.51	5.
34.67	14.40	4.15	13.54	29.20	4.99	6.37	5.
34.64	12.96	1.93	14.29	29.62	4.50	5.36	4.
34.12	11.74	2.98	11.98	29.50	4.47	6.13	4.
34.10	10.81	3.00	10.02	29.64	4.39	5.36	4.
34.05	9.64	3.09	7.71	29.82	4.45	5.56	3.
33.95	9.03	3.25	6.54	29.94	4.42	5.48	2.
33.93	8.42	3.27	5.55	30.22	4.40	5.16	2.
33.75	7.83	3.51	4.74	30.10	3.95	5.29	2.
33.60	7.55	3.68	4.26	29.90	3.79	5.42	1.
33.62	7.23	3.66	3.80	30.64	4.05	4.94	1.
33.53	6.94	3.75	3.43	30.43	4.03	4.98	1.
33.49	6.65	3.79	3.09	30.54	3.84	4.92	1.
33.41	6.37	3.37	2.81	30.36	3.15	5.02	1.
33.33	6.09	3.94	2.53	30.08	2.83	5.13	0.
33.27	5.86	4.00	2.32	30.12	2.59	5.12	0.

Figure 16

1.5 The case where K is unknown

All results reported so far have been for the case where K and α are known. In Sections 1.5 - 1.7, we explore the consequences of assuming that K, x, and v must all be estimated, with only α being known. We should anticipate that good estimates of x and v will now require less noisy measurements than before, since there is more estimating to be done. The principal object remains the comparison of the direct search and least squares procedures.

1.6 Direct Search Method (K unknown)

Rewrite (1) as $\bar{C}_i = Kd_i$ to emphasize that $\bar{C}_i$ is a linear function of K. So $d_i = |x + vt_i - x_i|^{-\alpha}$. The object now is still to minimize $E_n(x,v)$ as given by (2), but in the current notation

$$(6) \qquad E_n(x,v) = \sum_{i=1}^{n} (C_i - Kd_i)^2 = A_n - 2KB_n + K^2D_n ,$$

where

$$A_n = \sum_{i=1}^{n} C_i^2 ,$$

$$B_n = \sum_{i=1}^{n} C_i d_i , \text{ and}$$

$$D_n = \sum_{i=1}^{n} d_i^2$$

Even though it would be possible to minimize $E_n(x,v)$ by setting up a three dimensional grid on K, x, and v, it is easier to do the optimization on K analytically, since (6) is simply a quadratic expression in K. By equating the derivative to 0 and solving for K, we discover

$$K = B_n/D_n \quad (7)$$

Inserting this value for K back into (6), we obtain

$$E_n(x,v) = A_n - 2B_n^2/D_n + B_n^2/D_n = A_n - B_n^2/D_n \quad (7)$$

Furthermore, since A_n does not depend on x or v, minimizing $E_n(x,v)$ is the same as minimizing $1 - B_n^2/(A_n D_n)$, which is the same as maximizing

$$R_n(x,v) \equiv B_n^2/(A_n D_n) \quad (8)$$

Note that B_n and D_n depend on x and v in (8), although the notation suppresses that fact to avoid being too cumbersome. The factor A_n has been retained because $R_n(x,v)$ as defined in (8) is just the square of the correlation coefficient between the two sequences $C_1,\ldots,C_n$ and $d_1,\ldots,d_n$, which is a valuable interpretation. The LSQ procedure selects x and v to maximize $R_n(x,v)$; i.e., x and v are selected to make the theoretical and actual measurements as highly correlated as possible. The quantities A_n , B_n , and D_n are all sums, so those quantities are stored and updated at each measurement (see Sec. 1.9).

1.7 Least Squares Estimates (K unknown)

Our strategy here will be to eliminate K by sacrificing the first measurement. To simplify the required calculations, assume that $t_1 = x_1 = 0$; this can always be achieved by defining the origins of time and space to be at the first contact. From (1),

$$(9) \qquad (\bar{C}_i/\bar{C}_1)^{-1/\alpha} = |(x + vt_i - x_i)/x| = \bar{r}_i$$

Ignoring the absolute value (in other words, guessing direction), (9) becomes

$$(10) \qquad x(1 - \bar{r}_i) + vt_i - x_i = 0$$

If the measured value $r_i \equiv (C_i/C_1)^{-1/\alpha}$ is substituted for $\bar{r}_i$ in (10), the result is

$$(11) \qquad x(1 - r_i) + vt_i - x_i = \epsilon_i \ ,$$

where ϵ_i is an error. Since the left hand side is a linear expression in x and v , the task of minimizing $\sum_{i=1}^{n} \epsilon_i^2$

with x and v can be accomplished analytically, and we define the resulting estimates to be XLS1 and VLS1, respectively.

A different least squares estimate can be obtained if (10) is divided through by t_i when $i \geq 1$. The resulting sum of squares is $\sum_{i=1}^{n} \left(x \frac{1-r_i}{t_i} + v - \frac{x_i}{t_i} \right)^2$, and we define the minimizing x and v to be XLS2 and VLS2, respectively. The motiviation for dividing through by t_i is that the coefficient of one of the unknowns (namely v) becomes 1.0, which means that LS2 estimates are even easier to calculate than LS1 estimates. However, the LS2 and LS1 estimates are different, and the comparisons in the next section will show that the LS2 procedure does not make good use of measurements with high subscripts. The reason for this is that division by t_i gives the high subscript terms very little influence in the sum of squares.

1.8 Comparisons with K unknown

Figures 17-22 show a comparison of the LS1 and DIR procedures (we retain the name DIR even though K is estimated analytically) in the same circumstances and format as Figures 1-16. Performance is of course worse than in the case where K is known. Compare Figures 2 and 19, for example. For the same inputs, the amount of bias and especially the amount of variance are considerably smaller in the case where K is known. Alternatively, Figures 2 and 18 exhibit estimates of roughly the same quality where the amount of noise is 5 times as large in the case (Figure 2) where K is known. Knowledge of K is evidently worth a lot in terms of the amount of noise that the system can stand.

The comparison between the LS1 and DIR procedures leads to the same conclusions as when K is known. The DIR procedure seems to be superior statistically, particularly in the low variability of its estimates.

Figures 23-28 compare LS2 with DIR. Once again, DIR wins. Note that the variance of LS2 estimates declines slowly with the number of measurements; the reason for this was described earlier in 1.7. Compare Figures 18 and 24. LS1 and LS2 give the same estimates after 3 measurements, but the LS1 estimates improve much faster than the LS2 estimates as the number of measurements increases beyond 3. After 20 measurements, the LS2, LS1, and DIR procedures estimate x with sample standard deviations of 12.64, 3.74, and .78, respectively. These are substantial differences. Alternatively, compare Figures 23 and 24 at the $18^{\underline{th}}$ measurement. The LS2 procedure in Figure 23 has roughly the same mean and variance as the DIR procedure in Figure 24, but the noise level is 10 times as large in the latter. In this instance, employment of DIR vice LS2 is equivalent to a 10db gain in sensitivity.

1.9 An operational version of Direct Search

The principle drawback of the Direct Search procedure is the amount of time required to make an estimate. In order to get some idea of this time, a version of the Direct Search with unknown K procedure was written in BASIC for the TEK 4052 computer. The program updates the quantities A_n, B_n, and D_n used in (8) according to

$$
\begin{aligned}
A_{n+1} &= A_n + E_n C_n^2 \\
B_{n+1} &= B_n + E_n C_n d_n \\
D_{n+1} &= D_n + E_n d_n^2 , \quad \text{where} \\
E_n &= \exp(- (t_n - t_{n-1})/\tau)
\end{aligned} \tag{12}
$$

The reason for including the factor E_n in all of the updates is due to a conviction that some sort of discounting mechanism should be employed in practice to give more weight to recent than to past measurements, with τ being a "relaxation time" within which the parameters x and v might in practice vary a little bit. With this feature built in, the program takes about 40 seconds to compute, display, and select the maximum of the squared correlation coefficient R_n for $50 \times 30 = 1500$ grid points. Since R_n is always between 0 and 1, the displ is simply a 50×30 matrix of the first digit after the decimal point in R_n; an operational version might actually draw a contour map. It is felt that the display provides valuable information even after the maximizing grid point is known, since R_n can have multiple local maxima.

Figure 29 shows a sample map after processessing 3 observations. Note that the variables searched are speed and time late, rather than speed and distance. The only advantage of using time instead of distance is that the sign of time is necessarily positive. The program reports that the peak of the map is at time late 9.2 and speed -11, which is a point in the middle of the field of 9's in the upper right hand corner. The virtue of having the map is that an operator could see that there are also lots of 9's in the lower right hand corner; i.e., it is still possible that the speed is positive.

TRUE X= 30.0000 V= 5.0000 A1= 0.0 A2= 0.0001

XLS1	+/-	VLS1	+/-	XDIR	+/-	VDIR	+
0.0	0.0	0.0	0.0	18.64	7.94	4.61	1.
0.0	0.0	0.0	0.0	24.76	9.01	4.19	2.
29.98	1.74	5.03	0.48	30.18	1.74	4.98	0.
29.94	1.66	5.03	0.35	30.13	1.81	4.98	0.
29.82	1.25	5.04	0.34	30.12	1.32	4.95	0.
29.75	1.09	5.05	0.26	30.06	1.12	4.99	0.
29.80	0.90	5.05	0.24	30.00	0.94	5.02	0.
29.80	0.84	5.04	0.21	30.18	0.79	4.96	0.
29.86	0.85	5.03	0.22	30.16	0.88	4.96	0.
29.85	0.79	5.04	0.19	30.04	0.49	4.99	0.
29.85	0.77	5.03	0.20	30.04	0.49	4.99	0.
29.85	0.72	5.03	0.18	30.00	0.03	5.00	0.
29.84	0.71	5.04	0.18	30.00	0.03	5.00	0.
29.84	0.66	5.04	0.16	30.00	0.03	5.00	0.
29.86	0.66	5.03	0.17	30.00	0.03	5.00	0.
29.86	0.63	5.03	0.16	30.00	0.03	5.00	0.
29.87	0.64	5.03	0.16	30.00	0.03	5.00	0.
29.87	0.61	5.03	0.15	30.00	0.03	5.00	0.
29.88	0.62	5.03	0.15	30.00	0.03	5.00	0.
29.88	0.59	5.03	0.14	30.00	0.03	5.00	0.

Figure 17

TRUE X= 30.0000 V= 5.0000 A1= 0.0 A2= 0.0010

XLS1	+/-	VLS1	+/-	XDIR	+/-	VDIR	+
0.0	0.0	0.0	0.0	19.38	8.89	4.65	1.
0.0	0.0	0.0	0.0	17.86	11.10	5.77	3.
32.32	21.75	4.40	7.27	31.50	12.71	5.46	3.
27.80	20.90	5.13	4.07	32.32	12.55	5.00	2.
26.56	14.10	5.49	3.63	30.64	10.34	4.79	2.
21.75	6.43	6.46	1.42	30.96	9.48	4.35	1.
22.66	4.99	6.46	1.45	30.80	8.01	4.36	2.
21.72	4.42	6.58	1.16	31.56	7.64	4.64	1.
22.64	4.44	6.52	1.18	32.22	8.21	4.54	1.
22.17	4.39	6.59	1.06	31.34	6.76	4.74	1.
22.76	4.43	6.52	1.12	31.04	5.63	4.77	1.
22.35	3.84	6.60	0.92	30.78	4.57	4.34	1.
22.92	3.85	6.51	0.95	30.22	3.39	4.93	0.
22.73	3.79	6.54	0.90	30.02	3.15	4.99	0.
23.29	3.81	6.45	0.92	30.34	2.54	4.94	0.
23.19	3.79	6.46	0.90	30.48	2.19	4.90	0.
23.64	3.79	6.38	0.91	30.26	1.97	4.94	0.
23.52	3.76	6.40	0.88	30.06	1.58	4.99	0.
23.92	3.75	6.33	0.89	30.16	1.33	4.96	0.
23.83	3.74	6.35	0.87	30.16	0.78	4.96	0.

Figure 18

TRUE X= 30.0000 V= 5.0000 A1= 0.0 A2= 0.0050

XLS1	+/-	VLS1	+/-	XDIR	+/-	VDIR	+/-
0.0	0.0	0.0	0.0	21.36	11.56	4.97	2.70
0.0	0.0	0.0	0.0	6.94	8.93	5.70	5.34
-3.35	25.40	7.70	13.05	24.18	19.33	8.50	5.85
-0.10	9.47	9.17	4.57	19.42	18.00	6.99	5.40
1.39	7.00	10.92	2.14	25.18	16.34	7.06	4.35
2.18	4.15	9.98	0.84	28.02	16.42	6.22	4.16
2.59	3.57	10.71	0.39	28.64	15.37	6.07	4.45
2.40	2.97	10.43	0.69	29.52	14.05	5.32	3.84
2.49	3.49	10.33	0.35	31.92	16.02	5.12	3.67
2.54	3.15	10.54	0.70	33.02	14.94	4.79	3.30
2.79	3.43	10.34	0.79	32.62	14.33	4.58	3.37
2.92	2.85	10.70	0.62	32.72	13.54	4.57	2.96
3.26	2.99	10.79	0.68	32.00	12.54	4.53	2.99
3.29	2.81	10.71	0.62	32.14	11.82	4.52	2.75
3.53	2.95	10.78	0.65	32.60	11.12	4.41	2.68
3.55	2.81	10.73	0.61	32.28	9.73	4.49	2.24
3.79	2.95	10.76	0.65	33.72	10.18	4.18	2.37
3.83	2.82	10.72	0.62	32.38	9.01	4.50	2.08
4.08	2.96	10.73	0.66	31.88	9.00	4.57	2.10
4.10	2.86	10.69	0.63	31.64	6.48	4.64	1.43

Figure 19

TRUE X= 30.0000 V= 5.0000 A1= 0.0100 A2= 0.0

XLS1	+/-	VLS1	+/-	XDIR	+/-	VDIR	+/-
0.0	0.0	0.0	0.0	18.66	7.93	4.58	1.45
0.0	0.0	0.0	0.0	24.28	11.42	4.53	2.69
32.66	8.65	4.44	2.63	32.04	6.72	4.63	2.13
31.18	6.03	4.74	1.64	32.22	6.04	4.64	1.39
30.13	4.45	4.36	1.45	31.30	4.33	4.60	1.31
29.34	3.59	5.01	1.05	30.70	3.41	4.77	0.82
29.56	3.65	5.00	1.09	30.86	3.09	4.77	0.33
29.08	3.05	5.11	0.86	30.54	2.39	4.85	0.55
29.38	3.20	5.08	0.90	30.64	2.41	4.34	0.56
29.18	2.98	5.12	0.82	30.52	2.53	4.37	0.60
29.38	3.13	5.09	0.85	30.72	2.60	4.34	0.62
29.13	2.79	5.15	0.74	30.40	2.14	4.91	0.49
29.28	2.75	5.12	0.74	30.26	1.62	4.94	0.41
29.09	2.51	5.17	0.66	30.52	1.94	4.38	0.46
29.23	2.63	5.14	0.69	30.54	1.93	4.37	0.47
29.12	2.46	5.17	0.63	30.26	1.48	4.94	0.36
29.28	2.57	5.14	0.66	30.28	1.27	4.93	0.32
29.17	2.37	5.16	0.60	30.04	0.85	4.99	0.21
29.30	2.49	5.14	0.63	30.20	0.92	4.95	0.23
29.23	2.34	5.15	0.58	30.08	0.56	4.98	0.14

Figure 20

TRUE X= 30.0000 V= 5.0000 A1= 0.0500 A2= 0.0

XLS1	+/-	VLS1	+/-	XDIR	+/-	VDIR	
0.0	0.0	0.0	0.0	22.62	12.84	4.54	2
0.0	0.0	0.0	0.0	12.56	10.88	5.77	4
19.88	33.22	7.49	9.15	30.88	13.26	6.09	5
12.83	22.77	7.69	6.42	32.44	12.15	5.42	3
14.57	14.77	7.31	4.46	31.52	11.53	4.62	3
12.65	7.41	7.99	1.87	33.02	10.81	4.18	2
12.69	6.93	8.43	1.73	34.20	10.53	3.39	2
12.32	6.35	8.38	1.43	33.74	10.00	4.11	2
12.88	6.54	8.53	1.49	34.98	10.92	3.89	2
12.47	6.02	8.54	1.31	33.94	10.29	4.12	2
12.49	6.43	8.71	1.44	33.22	10.10	4.24	2
12.23	5.67	8.70	1.25	32.36	8.53	4.48	1
13.03	5.40	8.53	1.20	32.50	8.17	4.41	1
12.62	4.93	8.68	1.08	32.86	9.03	4.32	2
13.13	5.18	8.64	1.15	32.12	8.35	4.47	1
13.01	4.85	8.64	1.06	32.04	7.89	4.51	1
13.54	5.20	8.59	1.16	32.74	7.59	4.37	1
13.47	4.96	8.58	1.09	31.66	5.94	4.64	1
13.89	5.15	8.53	1.14	31.16	5.32	4.75	1
13.85	5.01	8.53	1.10	32.28	5.92	4.48	1

Figure 21

TRUE X= 30.0000 V= 5.0000 A1= 0.0100 A2= 0.0010

XLS1	+/-	VLS1	+/-	XDIR	+/-	VDIR	
0.0	0.0	0.0	0.0	19.66	10.25	4.70	1
0.0	0.0	0.0	0.0	17.30	13.08	6.07	3
31.64	22.44	3.95	8.07	31.74	12.54	5.53	3
24.69	16.38	5.64	3.10	32.78	12.42	4.95	2
24.55	8.55	5.90	2.44	31.10	10.44	4.66	2
21.32	6.37	6.49	1.48	30.82	8.96	4.83	1
22.23	5.39	6.51	1.50	31.28	8.21	4.72	2
21.27	5.26	6.64	1.33	31.80	8.33	4.55	2
22.25	5.48	6.58	1.38	32.60	8.70	4.44	2
21.70	5.22	6.67	1.22	32.40	8.19	4.50	1
22.30	5.23	6.61	1.28	31.60	6.94	4.63	1
21.82	4.56	6.70	1.04	31.06	5.63	4.77	1
22.37	4.42	6.62	1.05	30.78	4.55	4.80	1
22.15	4.32	6.65	0.99	30.52	4.33	4.87	0
22.69	4.31	6.57	1.01	30.82	3.84	4.82	0
22.57	4.26	6.59	0.98	30.88	3.40	4.31	0
23.02	4.22	6.51	0.99	31.00	3.09	4.78	0
22.88	4.19	6.54	0.96	30.18	1.89	4.97	0
23.30	4.18	6.46	0.97	30.24	2.13	4.95	0
23.21	4.18	6.48	0.96	30.44	1.70	4.90	0

Figure 22

TRUE X= 30.0000 V= 5.0000 A1= 0.0 A2= 0.0001

XLS2	+/-	VLS2	+/-	XDIR	+/-	VDIR	+/-
0.0	0.0	0.0	0.0	18.64	7.94	4.51	1.98
0.0	0.0	0.0	0.0	24.76	9.01	4.19	2.01
29.98	1.74	5.03	0.48	30.13	1.74	4.98	0.53
29.96	1.80	5.02	0.42	30.13	1.81	4.98	0.36
29.91	1.65	5.02	0.42	30.12	1.32	4.95	0.36
29.90	1.69	5.02	0.40	30.06	1.12	4.99	0.25
29.90	1.64	5.02	0.40	30.00	0.94	5.02	0.22
29.90	1.66	5.02	0.39	30.18	0.79	4.96	0.20
29.90	1.64	5.02	0.39	30.16	0.38	4.96	0.22
29.90	1.65	5.02	0.39	30.04	0.49	4.99	0.12
29.90	1.64	5.02	0.39	30.04	0.49	4.99	0.12
29.89	1.65	5.02	0.39	30.00	0.03	5.00	0.0
29.89	1.64	5.02	0.39	30.00	0.03	5.00	0.0
29.89	1.65	5.02	0.38	30.00	0.03	5.00	0.0
29.89	1.64	5.02	0.38	30.00	0.03	5.00	0.0
29.89	1.65	5.02	0.38	30.00	0.03	5.00	0.0
29.89	1.65	5.02	0.38	30.00	0.03	5.00	0.0
29.89	1.65	5.02	0.38	30.00	0.03	5.00	0.0
29.89	1.65	5.02	0.38	30.00	0.03	5.00	0.0
29.89	1.65	5.02	0.38	30.00	0.03	5.00	0.0

Figure 23

TRUE X= 30.0000 V= 5.0000 A1= 0.0 A2= 0.0010

XLS2	+/-	VLS2	+/-	XDIR	+/-	VDIR	+/-
0.0	0.0	0.0	0.0	19.38	8.39	4.55	1.63
0.0	0.0	0.0	0.0	17.86	11.10	5.77	3.90
32.32	21.75	4.40	7.27	31.50	12.71	5.46	3.34
29.66	25.00	4.57	5.11	32.32	12.55	5.00	2.66
29.42	16.61	4.75	4.20	30.64	10.34	4.79	2.73
27.79	14.94	5.10	3.47	30.96	9.48	4.35	1.97
27.86	13.58	5.13	3.39	30.80	8.01	4.36	2.04
27.39	13.28	5.24	3.18	31.56	7.64	4.64	1.82
27.41	13.12	5.28	3.17	32.22	8.21	4.54	1.97
27.20	13.09	5.34	3.09	31.34	6.76	4.74	1.50
27.22	12.96	5.36	3.09	31.04	5.68	4.77	1.38
27.03	12.83	5.42	2.99	30.78	4.57	4.34	1.02
27.02	12.77	5.44	3.00	30.22	3.39	4.93	0.82
26.90	12.72	5.48	2.95	30.02	3.15	4.99	0.71
26.91	12.68	5.49	2.95	30.34	2.54	4.94	0.63
26.83	12.67	5.52	2.92	30.48	2.19	4.90	0.52
26.83	12.65	5.53	2.93	30.26	1.97	4.94	0.48
26.76	12.65	5.55	2.91	30.06	1.58	4.99	0.38
26.77	12.64	5.56	2.91	30.16	1.38	4.96	0.34
26.71	12.64	5.58	2.90	30.16	0.78	4.96	0.20

Figure 24

TRUE X= 30.0000 V= 5.0000 A1= 0.0 A2= 0.0050

XLS2	+/-	VLS2	+/-	XDIR	+/-	VDIR	
0.0	0.0	0.0	0.0	21.36	11.56	4.97	2
0.0	0.0	0.0	0.0	6.94	8.93	5.70	5
-3.35	25.40	6.71	15.30	24.18	19.33	3.50	5
-2.29	16.92	8.58	7.34	19.42	13.00	6.99	5.
0.10	9.92	10.36	3.66	25.18	16.34	7.06	4
0.45	8.38	10.02	2.43	28.02	16.42	6.22	4.
0.48	8.33	10.55	2.27	28.64	15.37	6.07	4
0.55	7.90	10.49	1.97	29.52	14.05	5.32	3
0.63	7.82	10.73	1.91	31.92	16.02	5.12	3.
0.67	7.67	10.72	1.79	33.02	14.94	4.79	3
0.76	7.59	10.85	1.77	32.62	14.33	4.58	3.
0.80	7.44	10.84	1.68	32.72	13.54	4.57	2
0.86	7.43	10.93	1.69	32.00	12.54	4.53	2
0.88	7.36	10.93	1.65	32.14	11.82	4.52	2
0.92	7.36	11.00	1.65	32.60	11.12	4.41	2
0.93	7.32	11.00	1.63	32.28	9.73	4.49	2
0.96	7.33	11.06	1.64	33.72	10.18	4.18	2
0.97	7.30	11.06	1.62	32.38	9.01	4.50	2
0.99	7.31	11.10	1.63	31.88	9.00	4.57	2
1.00	7.29	11.11	1.61	31.64	6.48	4.64	1

Figure 25

TRUE X= 30.0000 V= 5.0000 A1= 0.0100 A2= 0.0

XLS2	+/-	VLS2	+/-	XDIR	+/-	VDIR	
0.0	0.0	0.0	0.0	18.66	7.93	4.68	1.
0.0	0.0	0.0	0.0	24.28	11.42	4.53	2.
32.66	8.65	4.44	2.63	32.04	6.72	4.63	2
32.30	8.36	4.50	2.24	32.22	5.04	4.64	1.
31.71	7.19	4.52	2.12	31.30	4.33	4.60	1.
31.60	7.24	4.54	1.99	30.70	3.41	4.77	0.
31.57	7.27	4.54	1.98	30.86	3.09	4.77	0.
31.52	7.26	4.57	1.90	30.54	2.39	4.85	0.
31.53	7.30	4.57	1.90	30.64	2.41	4.84	0.
31.52	7.35	4.58	1.87	30.52	2.53	4.37	0.
31.53	7.35	4.58	1.87	30.72	2.60	4.34	0.
31.52	7.39	4.59	1.84	30.40	2.14	4.91	0.
31.53	7.39	4.59	1.84	30.26	1.62	4.94	0.
31.52	7.42	4.60	1.83	30.52	1.94	4.38	0.
31.53	7.42	4.60	1.83	30.54	1.93	4.37	0.
31.53	7.45	4.60	1.82	30.26	1.48	4.94	0
31.54	7.46	4.60	1.82	30.28	1.27	4.93	0.
31.54	7.48	4.60	1.81	30.04	0.85	4.99	0.
31.54	7.49	4.60	1.81	30.20	0.92	4.95	0
31.54	7.51	4.61	1.80	30.08	0.56	4.98	0.

Figure 26

TRUE X= 30.0000 V= 5.0000 A1= 0.0500 A2= 0.0

XLS2	+/-	VLS2	+/-	XDIR	+/-	VDIR	+/-
0.0	0.0	0.0	0.0	22.62	12.84	4.54	2.2
0.0	0.0	0.0	0.0	12.56	10.33	5.77	4.9
19.88	33.22	7.49	9.15	30.88	13.26	6.09	5.0
16.37	23.37	6.99	6.34	32.44	12.15	5.42	3.9
15.48	20.66	7.97	7.26	31.52	11.53	4.62	3.3
12.79	19.87	8.06	4.69	33.02	10.81	4.18	2.6
12.56	18.58	8.28	4.60	34.20	10.53	3.39	2.7
12.29	18.00	8.29	4.26	33.74	10.00	4.11	2.4
12.00	17.89	8.47	4.27	34.93	10.92	3.89	2.6
11.71	17.70	8.54	4.12	33.94	10.29	4.12	2.3
11.53	17.89	8.67	4.19	33.22	10.10	4.24	2.4
11.35	17.54	8.72	4.04	32.36	8.53	4.48	1.9
11.35	17.48	8.78	4.04	32.50	8.17	4.41	1.9
11.10	17.27	8.84	3.94	32.86	9.03	4.32	2.1
11.07	17.30	8.90	3.96	32.12	8.35	4.47	1.97
10.97	17.20	8.92	3.91	32.04	7.39	4.51	1.82
10.93	17.28	8.97	3.94	32.74	7.59	4.37	1.79
10.85	17.18	8.99	3.89	31.66	5.94	4.64	1.32
10.81	17.20	9.03	3.91	31.16	5.32	4.75	1.21
10.76	17.13	9.05	3.87	32.28	5.92	4.48	1.33

Figure 27

TRUE X= 30.0000 V= 5.0000 A1= 0.0100 A2= 0.0010

XLS2	+/-	VLS2	+/-	XDIR	+/-	VDIR	+/-
0.0	0.0	0.0	0.0	19.66	10.25	4.70	1.06
0.0	0.0	0.0	0.0	17.30	13.08	6.07	3.38
31.64	22.44	2.97	9.86	31.74	12.54	5.53	3.44
28.16	20.90	4.71	4.65	32.78	12.42	4.95	2.76
27.94	15.22	4.99	3.81	31.10	10.44	4.66	2.71
26.60	14.54	5.26	3.35	30.82	8.96	4.83	1.97
26.73	13.55	5.30	3.33	31.28	8.21	4.72	2.13
26.14	13.36	5.44	3.13	31.80	8.33	4.55	2.03
26.24	13.33	5.48	3.15	32.60	8.70	4.44	2.09
26.01	13.26	5.55	3.06	32.40	8.19	4.50	1.81
26.08	13.09	5.57	3.06	31.60	6.94	4.63	1.65
25.91	13.00	5.62	2.98	31.06	5.63	4.77	1.27
25.92	12.93	5.64	2.99	30.78	4.55	4.80	1.08
25.79	12.87	5.68	2.94	30.52	4.33	4.87	0.99
25.82	12.84	5.70	2.95	30.82	3.84	4.82	0.91
25.73	12.81	5.73	2.92	30.88	3.40	4.81	0.77
25.74	12.79	5.74	2.93	31.00	3.09	4.78	0.72
25.67	12.78	5.77	2.91	30.18	1.89	4.97	0.43
25.68	12.76	5.78	2.91	30.24	2.13	4.95	0.50
25.63	12.76	5.79	2.90	30.44	1.70	4.90	0.39

Figure 28

```
--------------------TIME LATE-----------------==* 10
2356778888999999999999999999999999999999999999999999     -15
2356778889999999999999999999999999999999999999999999       -
2456778889999999999999999999999999999999999999999999       -
2456778899999999999999999999999999999999999999999999       -
2456778899999999999999999999999999999999999999999999       -
2456778889999999999999999999999999999999999999999999       -
2456778889999999999999999999999999999999999999999999       -
3466778889999999999999999999999999999999999999999999       -
4566778888899999999999999999999999999999999999999999       -
3134566777888888999999999999999999999999999999999999       -
2542012345566777788888888899999999999999999999999999       -
2356654210112334455666777778888888888899999999999999       -
1234567777654321100112222334445555666677777778888889       -
1223445667778888888877665443322211100011111222233334       -
1122333444455555666667777778888888889999999888888889
---------------------------------------------------- SPEED=0
1122333444555566666666666655554443322221111110000000001    -
1233455666554322110000111222223334444455555555666666       -
1234554321000112233344455555666666677777777777777788       -
2344310011233444555566666677777777777788888888888888       -
2431012234445556666677777777788888888888888888888888       -
3301233445566666777777777888888888888888888889999999       -
3012344556666777777788888888888888888899999999999999       -
1234455666677777778888888888888888889999999999999999       -
2344556666777777888888888888888899999999999999999999       -
3445566667777778888888888888889999999999999999999999       -
3445566677777788888888888899999999999999999999999999       -
3455566677777888888888888899999999999999999999999999       -
3455666777778888888888889999999999999999999999999999       -
3455666777788888888888899999999999999999999999999999       -
3455666777888888888889999999999999999999999999999999      15
PEAK OF MAP AT TIME LATE 9.2 SPEED  -11.   INPUT X AND T
```

Figure 29

Sample map of Squared Correlation Coefficient for 0 < time late < 10 and - 15 < speed < 15.

2. Maximum likelihood estimate for multiplicative noise.

2.1 Introduction

As was pointed out in Section 1.2 the estimate obtained by the direct search method has the virtue of being a maximum likelihood estimate in case of an additive white Gaussian noise. If a multiplicative noise is present, however, this is no longer the case. It is therefore of some interest to investigate to what extent the estimation may be improved by using the true maximum likelihood estimator. To do so it is of course necessary to make an assumption about the distribution of the multiplicative noise. We shall assume that the multiplicative noise is log-normal just as we have done in Section 1. Although at present there seems to be only a meager evidence in favor of this assumption, there is not much evidence in the contrary either. Thus, at present, our assumption can be perhaps justified as a convenient working hypothesis.

2.2 The model

We assume, as in Section 1, that the true range to the target at the time of the i-th measurement is

$$r_i = |u + vt_i - x_i| \text{ , } i = 1,\ldots,n \text{ ,} \tag{1}$$

where v and u are the speed and initial distance to a target. The position x_i of the sensor at the time t_i of the i-th measurement is assumed to be measured with respect to a reference direction along the target track with the initial position at $t_1 = 0$ of the sensor $x_1 = 0$ as the origin.

The target is assumed to travel with a constant speed and not be overtaken by the sensor during the n measurements. Thus, if the target travels in the reference direction the quantities u, v, t_i and $u + vt_i - x_i$ in (1) are all positive, while if the target travels against the reference direction the same effect is accomplished by changing the signs of all x_i's . Hence if we define for all $i = 1, \ldots, n$

$$\tilde{x}_i = \begin{cases} x_i & \text{if the target travels in the reference direction,} \\ -x_i & \text{if the target travels agains the reference direction,} \end{cases}$$

we can write for the range

$$\text{(2)} \qquad r_i = u + vt_i - \tilde{x}_i \ , \ i = 1, \ldots, n \ ,$$

with r_i , u , v , t_i all positive in either case. This eliminates the cumbersome absolute value in (1).

The i-th sensor measurement C_i is assumed related to the range r_i by the equation

$$\text{(3)} \qquad C_i = Kr_i^{-\alpha}Q_i \ , \ i = 1, \ldots, n \ ,$$

where $\alpha > 0$ is known constant, $K > 0$ is unknown constant signal strength, and $Q_1,\ldots,Q_n$ are independent, identically distributed positive random variables representing multiplicative measurement noise. The distribution of Q_i is assumed log-normal, i.e.

$$Q_i = e^{aW_i}$$

where a is a constant and W_i are standard Gaussian random variables. No additive noise is considered in this model.

2.3 Derivation of the maximum likelihood estimator

Let the random variables Q_i in (3) have the log-normal density

$$f_Q(q) = \frac{1}{\sigma\sqrt{2\pi}} \frac{1}{q} e^{-\frac{1}{2\sigma^2}\ln^2 q}, \quad q > 0, \tag{4}$$

where $\sigma > 0$ is a parameter.

Remark: This is equivalent to $Q_i = e^{A_1 W_i}$ with $\sigma^2 = e^{A_1^2}(e^{A_1^2} - 1)$ and W_i standard normal.

The likelihood function is from (3) and (4)

$$f(C_1,\ldots,C_n) = \prod_{i=1}^{n} \frac{r_i^{\alpha}}{K} f_Q\left(\frac{C_i r_i^{\alpha}}{K}\right) \tag{5}$$

$$= \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} \frac{1}{C_i} \exp\left[-\frac{1}{2\sigma^2} \ln^2 \frac{C_i r_i^{\alpha}}{K}\right],$$

and this is to be maximized with respect to u , v or u , v , K for K unknown subject to the constraints

$$r_i = u + vt_i - \tilde{x}_i \ , \ i = 1,\ldots,n \ .$$

Taking logarithm we get

$$(6) \qquad \ln f(C_1,\ldots,C_n) = -\, n \ln \sigma\sqrt{2\pi} - \sum_{i=1}^{n} \ln C_i - \frac{1}{2\sigma^2} \sum_{i=1}^{n} \ln^2 \frac{C_i r_i^{\alpha}}{K}$$

so that the problem is equivalent to minimizing

$$(7) \qquad \sum_{i=1}^{n} \ln^2 \frac{C_i r_i^{\alpha}}{K} \ .$$

If the signal strength K is known we thus conclude that the maximum likelihood estimate $\hat{u}$, $\hat{v}$ minimizes the objective function

$$(8) \qquad D(u,v) = \sum_{i=1}^{n} \left(\ln \frac{C_i r_i^{\alpha}}{K} \right)^2$$

subject to the constraints

$$r_i = u + vt_i - \tilde{x}_i \ , \ i = 1,\ldots,n \ .$$

For the case of unknown K we note that the function to be minimized

$$\sum_{i=1}^{n} \ln^2 \frac{C_i r_i^{\alpha}}{K} = \sum_{i=1}^{n} \left(\ln C_i r_i^{\alpha} - \ln K \right)^2$$

has as a function of $K > 0$ a single minimum at

$$\hat{K} = \left(\prod_{i=1}^{n} C_i r_i^{\alpha} \right)^{\frac{1}{n}} .$$

Upon substitution back into (7) we obtain the objective function

$$D(u,v) = \sum_{i=1}^{n} \left(\ln \overline{C}_i \overline{r}_i^{\alpha} \right)^2 , \tag{9}$$

where we have denoted

$$\overline{r}_i = r_i \left(\prod_{j=1}^{n} r_j \right)^{-\frac{1}{n}}$$

$$\overline{C}_i = C_i \left(\prod_{j=1}^{n} C_j \right)^{-\frac{1}{n}} , \quad i = 1,\ldots,n .$$

Thus the maximum likelihood estimate $\hat{u}$, $\hat{v}$ for the case of unknown signal strength K must minimize (9) again subject to the constraints

$$r_i = u + vt_i - \tilde{x}_i , \quad i = 1,\ldots,n .$$

Notice that in either case the estimator is still of a least-squares type but the dependence on the parameters u, v is nonlinear. Hence, for each set of observations $C_1, \ldots, C_n$ the maximum likelihood estimate $\hat{u}, \hat{v}$ must be determined by numerical minimization. In view of the nature of the problem considered in this report the simplest approach is to use the direct search method as in Section 1, applied this time to the objective function (8) or (9).

2.4 Results of computer simulation

Since the purpose was to investigate the possible improvement due to the use of maximum likelihood estimator simulation was performed with the same set of parameters as in Section 1.4 (for known K) and Section 1.8 (for unknown K). The data are displayed in the same format in Figures 30 to 35. For the case of known K, Figure 30 should be compared with Figure 9, Figure 31 with Figure 11, Figure 32 with Figure 12.

It is seen that indeed the maximum likelihood estimates result is smaller standard deviations and, especially for larger noise intensity, less bias.

Figure 33 was included to see how much would the performance deteriorate if the observation noise were in fact additive rather than multiplicative. Comparison with the XDIR and VDIR columns of Figure 5 shows that one must pay an estimation penalty for assuming multiplicative noise when the noise is actually additive. Nevertheless, the maximum likelihood estimator for the multiplicative noise still performs better than the least squares algorithm.

For the case of unknown signal strength K the results of simulation are in Figures 34 and 35 which should be compared with Figures 20 and 21 respectively. Here the performance of the maximum likelihood estimator is about the same as the direct search method although somewhat smaller bias can be detected in Figure 35.

In conclusion, if the noise were indeed multiplicative log-normal and if no additive noise were present then the maximum likelihood estimator would be generally preferable. However, the direct search method seems to be more robust and would probably be safer to use at least until a verifiable distributional assumptions can be made about the nature of the noises involved.

TRUE X= 30.0000 V= 5.0000 A1= 0.1000 A2= 0.0

XDIR	+/-	VDIR	+/-
30.26	2.09	3.45	1.57
29.64	2.00	7.18	6.61
29.54	2.08	7.08	5.95
30.04	2.09	4.99	4.57
29.98	2.20	5.11	3.49
30.18	2.37	4.70	3.28
30.06	2.19	4.98	2.70
30.08	1.90	4.99	2.11
30.06	1.94	4.99	1.82
30.22	1.62	4.77	1.36
29.96	1.61	5.10	1.26
29.96	1.43	5.12	0.96
30.02	1.32	5.02	0.83
30.10	1.32	4.97	0.81
30.04	1.26	4.96	0.63
30.06	1.21	5.02	0.56
30.18	1.29	4.95	0.57
30.12	1.28	4.95	0.52
30.22	1.42	4.91	0.55
30.18	1.16	4.94	0.43

Figure 30

TRUE X= 30.0000 V= 5.0000 A1= 0.5000 A2= 0.0

XDIR	+/-	VDIR	+/-
31.44	9.78	4.02	3.23
31.18	8.73	7.96	7.05
31.12	8.09	8.07	7.05
30.20	7.54	6.45	6.66
29.76	6.90	6.53	6.56
29.60	7.09	5.79	6.18
28.98	6.50	6.95	6.17
29.18	6.50	6.72	5.80
29.24	6.78	6.36	5.70
29.66	6.09	5.32	4.78
29.24	6.16	6.08	4.62
29.22	6.12	6.17	3.99
30.08	6.14	5.31	3.78
30.28	5.87	5.01	3.42
30.04	5.57	5.19	2.91
30.08	5.56	5.18	2.57
30.70	5.91	4.76	2.63
30.50	5.75	4.83	2.50
30.68	5.61	4.74	2.19
30.48	5.35	4.90	1.96

Figure 31

TRUE X= 30.0000 V= 5.0000 A1= 1.0000 A2= 0.0

XDIR	+/-	VDIR	+/-
31.80	14.43	5.83	5.06
32.36	13.56	8.46	7.14
32.20	12.64	8.46	7.03
30.58	12.02	7.10	6.83
29.92	11.05	7.25	7.05
29.38	10.61	6.87	6.88
29.00	10.19	7.72	6.55
29.26	10.15	7.68	6.70
28.98	10.08	7.10	6.66
29.60	9.44	5.80	5.98
29.04	9.44	7.03	5.70
29.00	9.51	7.17	5.49
29.76	9.34	6.05	5.15
30.08	9.54	5.56	5.22
29.40	9.40	5.97	4.82
29.76	9.40	5.35	4.34
30.14	9.54	5.30	4.31
30.10	9.57	5.23	4.14
30.34	9.10	4.92	3.77
30.56	9.08	4.94	3.43

Figure 32

TRUE X= 30.0000 V= 5.0000 A1= 0.0 A2= 0.0200

XDIR	+/-	VDIR	+/-
32.54	12.28	5.66	4.97
33.20	11.57	7.56	7.11
32.24	9.83	7.42	6.87
32.80	10.04	6.63	6.87
32.10	9.60	7.55	6.73
32.72	9.59	7.72	6.63
32.36	9.44	7.62	6.30
32.18	8.60	7.59	6.23
31.98	8.41	7.22	5.66
32.10	8.19	6.52	5.34
32.46	8.41	5.75	4.93
32.98	8.52	4.91	4.63
33.16	8.19	4.79	4.10
33.06	8.16	4.80	4.06
33.36	7.64	4.37	3.48
33.72	7.86	4.01	2.98
33.96	7.70	3.79	2.82
33.92	7.45	3.79	2.58
33.82	7.20	3.83	2.38
33.90	6.93	3.79	2.20

Figure 33

TRUE X= 30.0000 V= 5.0000 A1= 0.0100 A2= 0.0

XDIR	+/-	VDIR	+/-
50.00	0.04	13.72	2.36
36.38	10.28	6.38	3.44
30.56	5.29	5.04	1.46
31.74	5.47	4.58	1.15
31.48	4.60	4.61	1.14
31.12	3.72	4.75	0.78
31.16	3.16	4.72	0.73
30.96	2.99	4.81	0.62
30.62	2.73	4.86	0.62
30.80	2.55	4.79	0.55
31.14	2.41	4.74	0.54
30.68	2.35	4.85	0.50
30.66	1.96	4.85	0.45
30.70	2.00	4.84	0.45
30.52	1.72	4.88	0.41
30.26	1.51	4.94	0.36
30.24	1.18	4.94	0.29
30.24	0.86	4.94	0.22
30.20	0.92	4.95	0.23
29.92	0.39	5.02	0.10

Figure 34

TRUE X= 30.0000 V= 5.0000 A1= 0.0500 A2= 0.0

XDIR	+/-	VDIR	+/-
49.90	0.57	13.94	2.42
19.44	17.38	6.90	4.61
31.40	14.16	6.61	4.51
32.40	15.08	4.34	3.13
35.08	12.54	3.95	2.94
35.04	12.03	3.98	2.58
35.78	11.25	3.86	2.58
34.80	11.90	4.15	2.45
34.12	11.03	4.14	2.51
35.28	11.35	3.80	2.41
36.64	10.76	3.59	2.38
35.00	10.20	4.02	2.12
33.64	8.59	4.21	1.93
34.14	8.60	4.08	1.90
33.58	8.24	4.17	1.88
32.98	8.49	4.34	1.91
32.40	7.88	4.45	1.80
31.72	6.93	4.60	1.55
32.22	6.65	4.48	1.54
31.36	5.48	4.70	1.22

Figure 35

3. Determination of target direction.

3.1 Introduction

For the purpose of target tracking it is important to determine the direction of target motion after only few initial measurements. This should be done with as little knowledge of the signal parameters as possible. Specifically, the target speed, distance and the signal strength will typically be unknown at the beginning and cannot be reliably estimated from only a few measurements. In this section we propose a simple method for deciding the target direction. The results, however, are only preliminary. More investigation is needed to develop the method into a workable algorithm.

3.2 Description of the method

We assume here that the direction of the target is to be determined immediately after three initial measurements. We also assume that the locations of the two measurements following the initial one bracket the latter and that the coordinate axis is oriented towards the last measurement. (See Figure 36.)

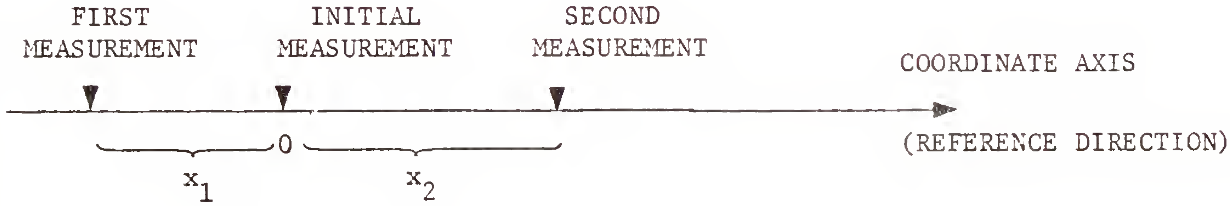

Figure 36

Thus the initial measurement is made at the coordinate $x_0 = 0$, the first at $-x_1 < 0$ and the second at $x_2 > 0$. Note that x_1 and x_2 are distances rather than coordinates, hence both are positive. The times are $t_0 = 0$, $t_1 > 0$ and $t_2 > t_1$ respectively. Let the initial distance to the target be $u > 0$ and the target speed $v > 0$, let r_j, $j = 0, 1, 2$, be the distance to the target at the time t_j. We then have

$$
\begin{aligned}
r_0 &= u \\
r_1 &= u + vt_1 + x_1 \\
r_2 &= u + vt_2 - x_2
\end{aligned}
$$

if the target travels in the reference direction, and

$$
\begin{aligned}
r_0 &= u \\
r_1 &= u + vt_1 - x_1 \\
r_2 &= u + vt_2 + x_2
\end{aligned}
$$

if the target travels against the reference direction.

If we look at the second difference

$$\Delta^2 r_0 = (r_2 - r_1) - (r_1 - r_0)$$

we see that it can be written as

$$\Delta^2 r_0 = v(t_2 - 2t_1) \mp (x_2 + 2x_1) .$$

Thus if $t_2 = 2t_1$ the second difference is positive or negative depending on whether the target travels against or along the reference direction respectively. Note that it is in fact sufficient that the term $v(t_2 - 2t_1)$ be small in magnitude relative to $x_2 + 2x_1$, a condition which can always be accomplished since the times t_1, t_2 and distances x_1, x_2 depend only on the motion of the observation platform.

Now the measurements C_0, C_1, C_2 are related to the distances to the target by

$$C_j = Kr_j^{-\alpha}Q_j + W_j \ , \qquad j = 0,1,2,$$

where K is the signal strength and Q_j's and W_j's are random variables representing the measurement noise. If the noise were absent ($Q_j = 1$ and $W_j = 0$) we could solve for r_j and calculate the second difference

$$\Delta^2 r_0 = K^{\frac{1}{\alpha}}\left(C_2^{-\frac{1}{\alpha}} - 2C_1^{-\frac{1}{\alpha}} + C_0^{-\frac{1}{\alpha}}\right) .$$

The direction of the target motion would then be determined from the sign of the expression

$$T = C_2^{-\frac{1}{\alpha}} - 2C_1^{-\frac{1}{\alpha}} + C_0^{-\frac{1}{\alpha}} .$$

In particular if $T < 0$ the target travels in the reference direction while if $T > 0$ the target travels against the reference direction. Notice that neither the signal strength K nor the true target speed v needs to be known to determine the direction, only the value of the exponent α is required.

In the presence of noise the second difference $\Delta^2 r_0$ is no longer proportional to the test statistic T but we can still hope that T will do the job. In fact, if $W_j = 0$ and $E\left[Q_j^{1/\alpha}\right] = 1$ then T is a point estimator for $K^{-1/\alpha}\Delta^2 r_0$ obtained by the method of moments. If we make a natural assumption that the mean of the multiplicative noise equals 1 and if the noise Q_j is log-normal then $E\left[Q_j^{1/\alpha}\right] = (E\left[Q_j\right])^{1/\alpha^2}$ is indeed one.

The next section describes the results of a simulation experiment designed to test the usefulness of the statistic T.

3.3 Computer simulation

In order to gain some appreciation of the performance of the method we again used computer simulation. The parameters were the same as in Section 1, true range 30NM, target speed 5KT, signal strength $K = 1$ and the exponent $\alpha = 1$. For each simulation run 1000 samples of the first three observations C_0, C_1, C_2 were generated. Each sample was generated both for positive true target direction (along the reference axis) and for negative true direction (against the reference axis.) Errors in direction determination were counted and the counts were used to estimate the error probabilities for either of the true directions.

For the first five runs we used the noise model of Section 1, equation (5), i.e. multiplicative log-normal noise and additive Gaussian noise. This was done mainly for the sake of comparison with earlier simulation results in this report. The results are summarized in TABLE 1. It is seen that for moderate values of the coefficients A_1 and A_2 the error probabilities are small enough for the method to be useful. For larger A_1 and A_2, however, the direction determination tends to be random, which is not too surprising with only three observations.

For the next series of simulation runs the observations were generated using the equation

$$C_j = K(1 + Z_j)r_j^{-\alpha} , \qquad j = 0, 1, 2,$$

where Z_0, Z_1, Z_2 were independent random variables with the density

$$f_Z(z) = \frac{1 + \beta}{2} (1 - |z|^{\beta}) , \qquad |z| < 1 .$$

The parameter $\beta > 0$ was varied from about 300 to zero. Note that the density function is symmetric about zero and as β decreases it flattens towards the uniform density on (-1, 1). Thus the multiplicative noise factor $1 + Z_j$ always averages to 1 but, as β decreases, tends to distort each true measurement $Kr_j^{-\alpha}$ by larger and larger percentage, rendering it almost useless for $\beta = 0$ (uniform distribution).

The estimated error probabilities are shown in Figure 37. On the abscissa of the graph is the ratio of the noise standard deviation $\sigma(1 + Z_j)$ to that of the uniform distribution on $(-1, 1)$, viz. $\sqrt{3}$, which seems to be more informative than the parameter β itself. Of course, β can be easily recovered since

$$\sigma^2(1 + Z_j) = \frac{2}{(\beta + 2)(\beta + 3)}$$

TRUE DIRECTION:		POSITIVE	NEGATIVE
A1	A2	ERROR PROBABILITY	
.001	.001	.00	.00
.01	.001	.01	.00
.05	.001	.05	.06
.1	.001	.27	.22
.5	.001	.47	.43

TABLE 1.

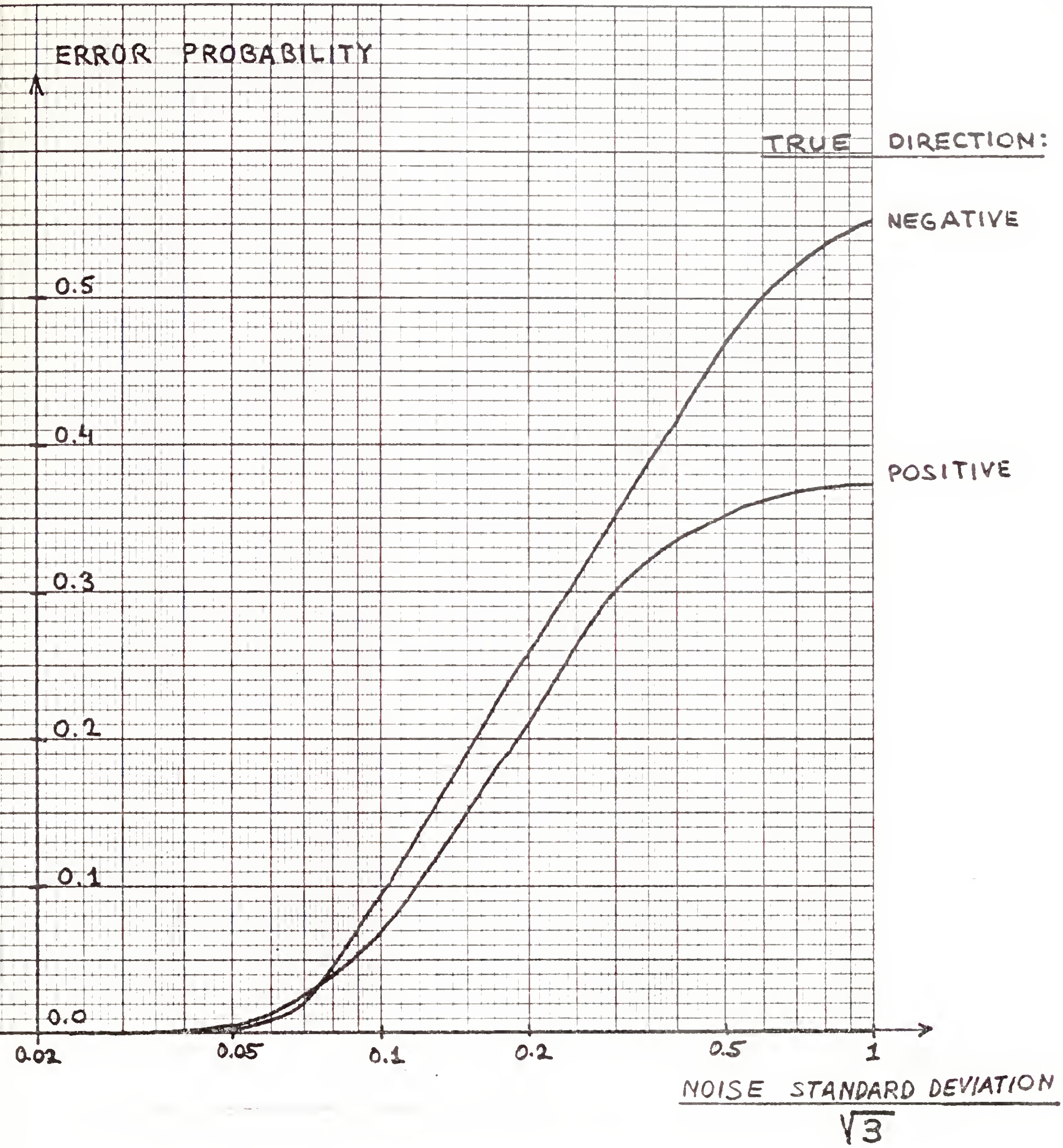
ERROR PROBABILITY
TRUE DIRECTION:
NEGATIVE
POSITIVE
0.5
0.4
0.3
0.2
0.1
0.0
0.02
0.05
0.1
0.2
0.5
1
NOISE STANDARD DEVIATION
√3

Figure 37

Notice that for the problem parameters used here there is a bias towards the positive direction. However, the results do indicate that the method may lead to a practical direction determination algorithm.

4. Equivalent sweep width.

In this section we consider the two dimensional problem of detecting a target that will be detected if either

a) the searcher comes within r of the target in any direction.

b) the searcher crosses a line segment of length L that the target trails behind it, where $L > r$.

The situation is illustrated in Figure 1. The angle θ is the angle perpendicular to the searcher's course in target centered coordinates. The heavy line of length $w(\theta)$ is the sweep width in the sense that detection will occur if and only if the searcher's track crosses it. The formula for $w(\theta)$ is

$$(1) \qquad w(\theta) = \begin{cases} r + L\sin\theta & \theta > \theta_0 \equiv \arcsin\left(\frac{r}{L}\right) \\ \\ 2r & \text{if} \quad \theta < \theta_0 \end{cases}$$

There are two assumptions commonly made about θ : one is that θ is uniformly random (area search), and the other is that θ is determined by relative speed in a barrier search. These two cases are analyzed below.

4.1 Area search

If θ is completely random, the average value of $w(\theta)$ can be found by assuming θ to be uniformly random in the first quadrant:

$$w = \frac{2}{\pi}\left[\int_0^{\theta_0} 2r d\theta + \int_{\theta_0}^{\pi/2} (r + L\sin\theta) d\theta\right]$$

$$= \frac{2}{\pi}\left[2r\theta_0 + r(\pi/2 - \theta_0) + L\cos\theta_0\right]$$

(2) $$= r + \frac{2}{\pi}\left[r\theta_0 + \sqrt{L^2 - r^2}\right]$$

when $r = 0$, the equivalent sweep width is $(2/\pi)L$, a familiar result. The point is that there is also an equivalent sweep width in the case where $r \neq 0$.

4.2 Barrier Search (back and forth perpendicular to target tracks

In this case θ is $\arcsin v/\sqrt{v^2 + u^2}$, where u and v are the target and searcher speeds, respectively. Let the result of substituting this into (1) be W_v, since we wish to emphasize the effect of searcher speed:

(3) $$W_v = \begin{cases} r + Lv/\sqrt{v^2 + u^2} & \text{if} \quad r/L < v/\sqrt{v^2 + u^2} \\ 2r & \text{if} \quad r/L > v/\sqrt{v^2 + u^2} \end{cases}$$

The probability of detection when the searcher must cover a length of barrier B is [reference 1]:

(4) $$P_d = \min\left\{1, \frac{W_v}{B}\sqrt{1 + v^2/u^2}\right\}$$

For example, Figure 2 shows a graph of P_d vs. v/u in the case $r = 1$, $L = 2$, $B = 10$, obtained by substituting (3) into (4). The point is that P_d is a very strong function of v/u over the region where $P_d < 1$ — the curve is actually convex. This strong dependence on speed is because sweep width increases with speed due to an increasingly favorable geometry, so the all important product of v and W_v increases fast. In practice, increasing speed will also have a detrimental effect on signal-to-noise ratio, so the "always proceed at top speed" conclusion indicated by this simple analysis does not hold. Nonetheless, the geometrical considerations are important. We claim, for example, that the best speed in a barrier patrol is greater than the best speed in area search, since W in (2) does not depend on v.

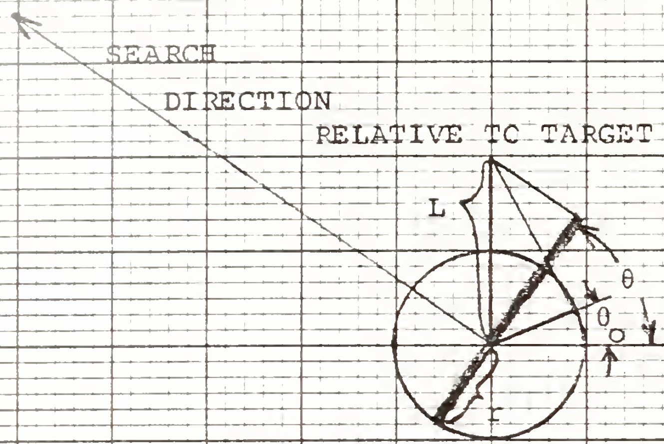

Figure 1

Heavy line is $w(\theta) = r + L \sin\theta$

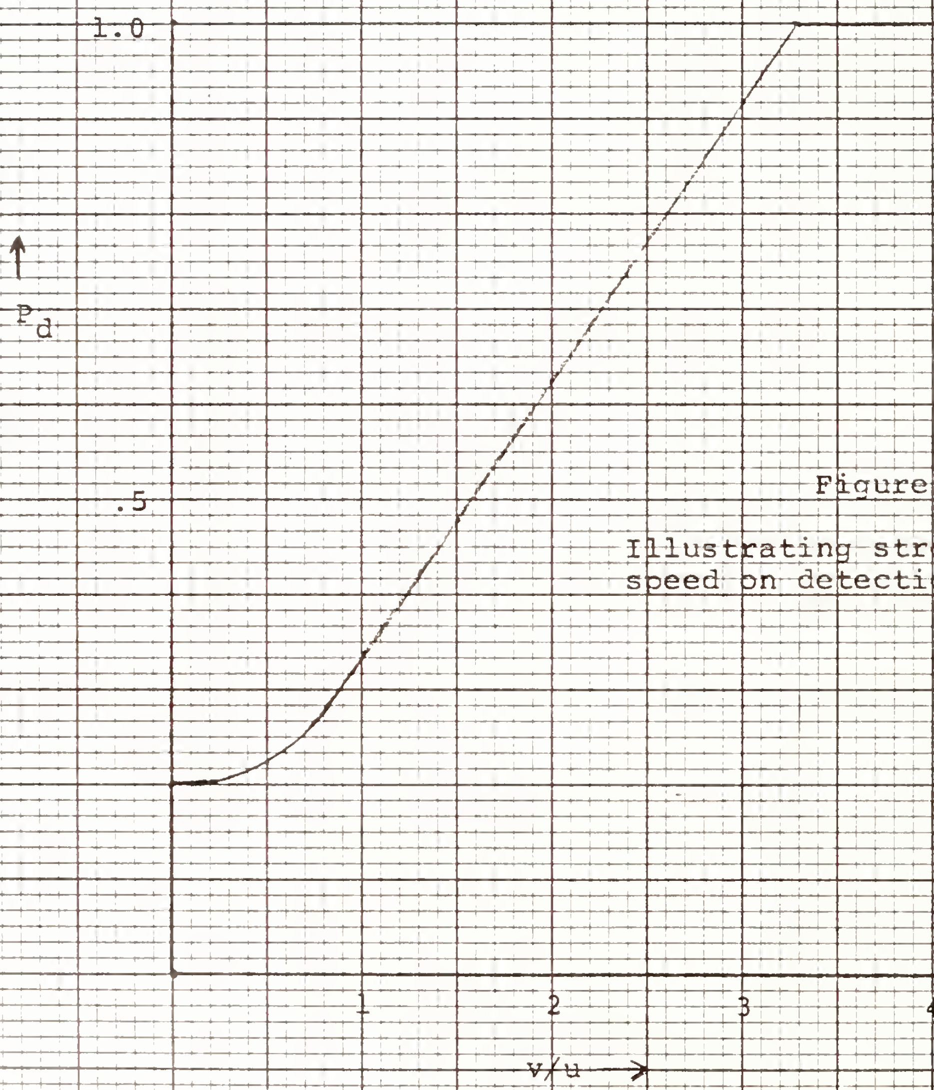

Figure 2

Illustrating strong effect of speed on detection probability

References

[1] Washburn, A. R. (1981), Search and Detection, sec. 1.3, Operations Research Society (MAS section).

DISTRIBUTION LIST

	NO. OF COPIES
Office of Research Administration Code 012A Naval Postgraduate School Monterey, CA 93940	1
Library, Code 0142 Naval Postgraduate School Monterey, CA 93940	2
Naval Underwater Systems Center Newport RI 02840 Attn: Code 3633	1
General Physics Corp. 2251 San Diego Ave. Suite B-110 San Diego, CA 92110 Attn: Joe Kronzer	1
Interstate Electronics Corp. 1001 East Ball Rd. Anaheim, CA 92803 Attn: Russ Roberts	1
Don Wagner, Ass. Station Square One Paoli, PA 19301 Attn: Barry Belkin	1
Assoc. Prof. A. R. Washburn Code 55Ws Naval Postgraduate School Monterey, CA 93940	7
Assoc. Prof. B. O. Shubert Code 55Sy Naval Postgraduate School Monterey, CA 93940	5